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Finite Element Analysis of the Stability of Fluid Motions

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The linear and global theories of stability of buoyancy-driven flows are reviewed, and corresponding eigenvalue problems whose eigenvalues give critical values of Rayleigh number are formulated in variational form. Penalty-finite element approximations of these problems are constructed. After a discussion of the properties of the solutions, the formulation is applied to the problem of stability of a container of fluid containing internal heat sources and heated from below.

1. INTRODUCTION

We are concerned here with the problem of determining the conditions under which solutions to problems of fluid flow are stable. The conditions we have in mind are upper bounds on one or more parameters of the system. More particularly, the problem under consideration is that of buoyancy-driven flow of viscous, incompressible heat-conducting fluids, and it is assumed that the Oberbeck–Boussinesq approximations applied to the Navier–Stokes equations give a reasonable model. Under these circumstances the parameter of interest is primarily the Rayleigh number, and we will accordingly seek bounds on the Rayleigh number. These bounds will appear as eigenvalues of appropriate eigenvalue problems, and one of our aims is to formulate and implement the variational theory for these eigenvalue problems and to seek finite element approximations of the solutions.

The problem has received much attention in the literature, the majority of investigations concentrating on analytical methods for determining critical values of Rayleigh number. The methods used can be divided conveniently into two categories: the linear theory of stability, due essentially to Poincaré [16], and the energy or global theory of stability, due to Serrin [18, 19] for the Navier–Stokes

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equations, and extended by Joseph [10, 11] and by Shir and Joseph [20] to the problem at hand. The linear theory gives sufficient conditions for instability, while the global theory gives sufficient conditions for stability. The two theories lead to two generally distinct eigenvalue problems, the least eigenvalues corresponding to the critical values of Rayleigh number. One problem for which the two theories coincide is the Bénard problem, which has been extensively documented and discussed (see Chandrasekhar [4], Joseph [13]). This is the problem of stability of a stationary fluid layer heated from below; at a critical value of Rayleigh number the layer loses its stability and cellular motion sets in.

Recently, Galdi and Straughan [7] have re-examined the connection between linear and global stability, and have applied their theory to the problem of gravity-dependent motion of a suspension of swimming micro-organisms. Another area in which linear and energy stability theory has found application is that of magnetohydrodynamics; a recent contribution which includes a good review of the work in this area is that of Galdi [6].

In Section 2 we formulate the eigenvalue problems corresponding to the linear and global theories and derive variational statements of these problems. We pay full attention to the question of the choice of spaces in which solutions are sought and make use of techniques which are by now standard to derive variational eigenvalue problems. The well-known penalty method is used to remove the hydrostatic pressure as a variable, so that solutions are obtained, in effect, for fluids with small compressibility.

Finite element approximations of the penalised problems are constructed and, after a discussion of the convergence of these approximations, numerical results are presented. We focus attention on the problem of a layer of fluid of finite extent, heated from below, with internal heat sources present. Existing analytical results for the Bénard problem (no internal heat sources) assume either an unbounded domain (Chandrasekhar [4], Sparrow, Goldstein, and Johnsson [21]) or a bounded domain with unrealistic boundary conditions on the horizontal surfaces (Hall and Walton [8]); numerical investigations by the finite element method have been carried out for bounded domains by Jackson and Winters [9], who use the method of parameter stepping and bifurcation search, and by Cliffe and Winters [3], who use an algorithm which locates symmetry-breaking bifurcation points. Our results for the Bénard problem represent an extension of all of these results in that we are able to elucidate the behaviour of critical Rayleigh number as a function of width–depth ratio for a wide range of ratios and for a variety of boundary conditions. Subsequently, we examine the situation for which internal heat sources give rise to a quadratic temperature distribution within the fluid. Global stability for this problem has been examined by Joseph and Shir [14], and linear stability by Sparrow *et al.* [21], for the case of an unbounded layer. Once again we show how the critical Rayleigh number varies, both with the intensity of the internal heat source and with the horizontal extent of the layer.

An analysis of the convergence of penalty-finite element approximation for the global stability problem has been discussed in a recent contribution (Reddy [17]).

2. Variational Eigenvalue Problems for Fluid Stability

We are concerned with the stability of motion of a viscous, incompressible heat-conducting fluid which occupies a bounded domain $\Omega \subset \mathbb{R}^N$ ($N \leqslant 3$). The motion of the fluid is assumed to be modelled adequately when the Oberbeck–Boussinesq assumptions are applied to the Navier-Stokes equations for heat-conducting, buoyancy-driven flow [13]. The resulting set of equations then comprises the equation of momentum balance

$$\rho_0\left(\frac{\partial \mathbf{U}}{\partial t}+(\nabla \mathbf{U})\,\mathbf{U}\right)=\operatorname{div}\mathbf{S}+\rho_0(1-\alpha(T-T_0))\,\mathbf{g}; \tag{2.1}$$

the equation of heat conduction

$$\frac{\partial T}{\partial t}+\mathbf{U}\cdot\nabla T=\kappa\nabla^2 T+Q; \tag{2.2}$$

the incompressibility condition

$$\operatorname{div}\mathbf{U}=0. \tag{2.3}$$

In (2.1)–(2.3), $\mathbf{U}$ is the velocity, T the temperature, $\mathbf{g}$ the gravity vector, ρ_0 the mass density at reference temperature T_0, Q a prescribed heat source field, and κ the thermal diffusivity. The stress tensor $\mathbf{S}$ is given as a function of velocity and pressure by

$$\mathbf{S}(\mathbf{U}, P)=-P\mathbf{I}+2\mu\,\mathbf{D}(\mathbf{U}), \tag{2.4}$$

where P is hydrostatic pressure, μ is the dynamic viscosity, and

$$\mathbf{D}(\mathbf{U})=\tfrac{1}{2}(\nabla\mathbf{U}+\nabla^T\mathbf{U}) \tag{2.5}$$

is the deformation rate tensor.

Equations (2.1)–(2.5) are required to hold on Ω. Boundary conditions on the boundary Γ of Ω are

(i) *temperatures*: prescribed temperature

$$T=T_0 \qquad \text{on } \Gamma_T, \tag{2.6}$$

prescribed heat flux

$$\partial T/\partial n=q_0 \qquad \text{on } \Gamma_q, \tag{2.7}$$

mixed condition

$$\partial T/\partial n+kT=r_0 \qquad \text{on } \Gamma_k, \tag{2.8}$$

where $\Gamma_T \cup \Gamma_q \cup \Gamma_k = \Gamma$ and $\Gamma_T, \Gamma_q, \Gamma_k$ are mutually disjoint, and $\mathbf{n}$ is the outward unit normal vector. The scalar function k is known as the Biot number.

(ii) *velocity*: prescribed velocity

$$\mathbf{U} = \mathbf{U}_0 \qquad \text{on } \Gamma_U, \tag{2.9}$$

prescribed tangential surface traction and normal velocity

$$\mathbf{t} \cdot \mathbf{Sn} = S_t, \qquad \mathbf{U} \cdot \mathbf{n} = U_n \qquad \text{on } \Gamma - \Gamma_U. \tag{2.10}$$

The velocity-traction set of boundary conditions (2.10) is not the most general possible, but the combination given above will suffice for our needs.

Initial conditions are

$$\mathbf{U}(\mathbf{x}, 0) = \mathbf{U}^0(\mathbf{x}), \qquad \mathbf{T}(\mathbf{x}, 0) = \mathbf{T}^0(\mathbf{x}) \qquad \text{in } \Omega. \tag{2.11}$$

Our main interest is to investigate the stability of a solution $(\mathbf{U}, T, P)$ of (2.1)–(2.11). The general procedure will be to subject this system to an arbitrary disturbance and to determine the conditions under which the disturbance will decay with time. Such conditions are normally expressed as bounds on the parameters for the problem, as will be seen.

Suppose then that the initial conditions are disturbed so that the solution to (2.1)–(2.11) is now

$$\mathbf{U} + \mathbf{u}, \qquad T + \theta, \qquad P + p, \tag{2.12}$$

where $(\mathbf{u}, \theta, p)$ constitutes the disturbance. By substituting (2.12) in (2.1)–(2.11) and subtracting from the resulting set of equations the set describing the basic motion, we arrive at the set of equations for the disturbance. Before displaying this set of equations we first non-dimensionalise by dividing

$$\mathbf{U}, \qquad \mathbf{D}(\mathbf{U}), \qquad T, \qquad \mathbf{g}, \qquad \mathbf{x}, \qquad t$$

by

$$U', \qquad U'/d, \qquad T', \qquad |\mathbf{g}|, \qquad d, \qquad d^2/\nu,$$

and the disturbance

$$\mathbf{u}, \qquad \mathbf{D}(\mathbf{u}), \qquad \theta, \qquad p$$

by

$$\nu/d, \qquad \nu/d^2, \qquad \left(\frac{\nu^3 T'}{g d^3 \kappa \alpha}\right)^{1/2}, \qquad \frac{\rho_0 \nu^2}{d^2};$$

here U', T', and d are characteristic magnitudes of velocity, temperature and length, and $\nu = \mu/\rho_0$ is the kinematic viscosity. The notation used for dimensionless variables is the same as that for the dimensional counterparts.

The introduction of the non-dimensionalisation into the equations for the disturbance now yields the set of equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathrm{Re}((\nabla \mathbf{U})\,\mathbf{u} + (\nabla \mathbf{u})\,\mathbf{U}) + (\nabla \mathbf{u})\,\mathbf{u} = -\mathscr{R}\theta \mathbf{g} - \nabla p + 2\,\mathrm{div}\,\mathbf{D}(\mathbf{u}), \tag{2.13}$$

$$\mathrm{Pr}\left(\frac{\partial \theta}{\partial t} + \mathrm{Re}\,\nabla\theta \cdot \mathbf{U} + \nabla\theta \cdot \mathbf{u}\right) + \mathscr{R}\nabla T \cdot \mathbf{u} = \nabla^2 \theta, \tag{2.14}$$

$$\mathrm{div}\,\mathbf{u} = 0 \tag{2.15}$$

on Ω, and

$$\mathbf{u} = \mathbf{0} \text{ on } \Gamma_U, \qquad \mathbf{t} \cdot \mathbf{S}(\mathbf{u}, p)\,\mathbf{n} = 0, \qquad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma - \Gamma_U, \tag{2.16}$$

$$\theta = 0 \text{ on } \Gamma_T, \qquad \nabla\theta \cdot \mathbf{n} = 0 \text{ on } \Gamma_q, \qquad \nabla\theta \cdot \mathbf{n} + k\theta = 0 \text{ on } \Gamma_k, \tag{2.17}$$

and the initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}), \qquad \theta(\mathbf{x}, 0) = \theta^0(\mathbf{x}) \qquad \text{in } \Omega. \tag{2.18}$$

The parameters which appear in these equations are the Rayleigh number $\mathscr{R}$, Reynolds number Re, and Prandtl number Pr, defined by

$$\mathscr{R} = \sqrt{\frac{\alpha g T' d^3}{\nu \kappa}}, \qquad \mathrm{Re} = \frac{U' l}{d}, \qquad \mathrm{Pr} = \frac{\nu}{\kappa}. \tag{2.19}$$

Equations (2.13)–(2.17) form the basis for the study of stability of the basic state. There are essentially two procedures for dealing with these equations: in the linear theory of stability, the disturbance is assumed small enough to justify the retention only of *linear* terms in ($\mathbf{u}$, θ, p), while in the global theory of stability the size of the disturbance is not restricted.

Linear Stability

Equations (2.13) and (2.14) are linearised by removing the terms $(\nabla \mathbf{u})\,\mathbf{u}$ and $\nabla\theta \cdot \mathbf{u}$; we then seek a solution of the resulting linear homogeneous set of equations of the form

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t) &= \mathbf{u}(\mathbf{x}) \exp(\sigma t) \\ \theta(\mathbf{x}, t) &= \theta(\mathbf{x}) \exp(\sigma t) \\ p(\mathbf{x}, t) &= p(\mathbf{x}) \exp(\sigma t), \end{aligned} \tag{2.20}$$

where σ is in general complex. Substitution in (2.13)–(2.17) leads to the equations

$$\begin{gathered}\sigma\mathbf{u} + \mathrm{Re}((\nabla\mathbf{U})\,\mathbf{u} + (\nabla\mathbf{u})\,\mathbf{U}) = -\mathscr{R}\theta\mathbf{g} - \nabla p + 2\,\mathrm{div}\,\mathbf{D}(\mathbf{u}),\\ \mathrm{Pr}(\sigma\theta + \mathrm{Re}\,\nabla\theta\cdot\mathbf{U} + \nabla\theta\cdot\mathbf{u}) + \mathscr{R}\nabla T\cdot\mathbf{u} = \nabla^2\theta,\\ \mathrm{div}\,\mathbf{u} = 0,\end{gathered} \tag{2.21}$$

together with boundary conditions.

The growth or decay of the disturbance depends on the sign of the real part of σ. If $\mathrm{re}(\sigma) > 0$ the disturbance grows with time, whereas if $\mathrm{re}(\sigma) < 0$ the disturbance decays and the basic flow is stable. The flow is neutrally stable if $\mathrm{re}(\sigma) = 0$. States of marginal stability can be one of two kinds: small disturbances can grow (or decay) aperiodically, or they can grow (or decay) with oscillations of increasing (or decreasing) amplitude. In the former case $\mathrm{im}(\sigma) = 0$ and the principle of exchange of stabilities is said to hold (Joseph [12, 13], Galdi and Straughan [7]). In the latter case $\mathrm{im}(\sigma) \neq 0$. As is customary in investigations of this kind we assume that the principle of exchange of stabilities holds. We further assume that the Reynolds and Rayleigh numbers are related by $\mathrm{Re} = \beta\mathscr{R}$, where β is a known constant; then for given $\mathscr{R}$ we can determine the smallest value of σ for which (2.21) hold. The critical Rayleigh number of linear theory, $\mathscr{R}_{\mathrm{L}}$, is the value of $\mathscr{R}$ for which $\mathrm{re}(\sigma) = 0$ ($\mathrm{im}(\sigma) = 0$ also), so by setting $\sigma = 0$ in (2.21) we have an eigenvalue problem for $\mathscr{R}_{\mathrm{L}}$: find $\mathbf{u}, \theta, p$, and $\mathscr{R}_L$ such that

$$\begin{gathered}2\,\mathrm{div}\,\mathbf{D}(\mathbf{u}) - \nabla p = \mathscr{R}_{\mathrm{L}}(\theta\mathbf{g} + \beta(\nabla\mathbf{U})\,\mathbf{u} + \beta(\nabla\mathbf{u})\,\mathbf{U}),\\ \nabla^2\theta = \mathscr{R}_{\mathrm{L}}(\nabla T\cdot\mathbf{u} + \beta\nabla\theta\cdot\mathbf{U}),\\ \mathrm{div}\,\mathbf{u} = 0,\end{gathered} \tag{2.22}$$

in Ω, and boundary conditions (2.16)–(2.17). Since the linear theory makes no prediction about the effects of large disturbances, clearly it gives only a *sufficient* condition for *instability*: if $\mathscr{R} > \mathscr{R}_{\mathrm{L}}$, then the basic flow is unstable. Flows for which $\mathscr{R} < \mathscr{R}_{\mathrm{L}}$, though judged stable by the linear theory, may be unstable to sufficiently large disturbances.

With a view to constructing Galerkin approximations of this problem we reformulate (2.22) in weak or variational form. First we define the spaces

$$V = \{\mathbf{v} = (v_1, ..., v_N): v_i \in H^1(\Omega),\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_U,\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma - \Gamma_U\},$$

$$Q = \{\phi \in H^1(\Omega): \phi = 0 \text{ on } \Gamma_T\}.$$

V and Q are closed subspaces of $(H^1(\Omega))^N$ and $H^1(\Omega)$ and are, consequently, Hilbert spaces with inner products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_Q$ defined by

$$(\mathbf{u}, \mathbf{v})_V = \int_\Omega \mathbf{u}\cdot\mathbf{v} + \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx,$$

$$(\phi, \theta)_Q = \int_\Omega \theta\phi + \nabla\theta\cdot\nabla\phi\,dx.$$

It is also convenient to define the product space $\bar{V} = V \times Q$, which is a Hilbert space with inner product $(\cdot, \cdot)_{\bar{V}}$ defined by

$$(\bar{u}, \bar{v})_{\bar{V}} = (\mathbf{u}, \mathbf{v})_V + (\theta, \phi)_Q,$$

where $\bar{u} = (\mathbf{u}, \theta)$ and $\bar{v} = (\mathbf{v}, \phi)$. We assume for definiteness that $k \in L_2(\Gamma_k)$, $\mathbf{g} \in (L_2(\Omega))^N$, $T \in H^1(\Omega)$, and that $\mathbf{U} \in (H^1(\Omega))^N$.

The pressure p may be eliminated as an unknown by introducing the penalty approximation (see, for example, Carey and Oden [2]): we dispense with $(2.22)_3$, replace p by $p_\varepsilon = -\varepsilon^{-1} \operatorname{div} u_\varepsilon$ and formulate the following *penalised variational eigenvalue problem*: find $\bar{u}_\varepsilon \in \bar{V}$ and $\mathscr{R}_{L\varepsilon} \in \mathbb{R}$ such that

$$\begin{aligned} a(\mathbf{u}_\varepsilon, \mathbf{v}) + \varepsilon^{-1}(\operatorname{div} \mathbf{u}_\varepsilon, \operatorname{div} \mathbf{v}) &= \mathscr{R}_{L\varepsilon}\{c(\theta_\varepsilon, \mathbf{v}) + e(\mathbf{u}_\varepsilon, \mathbf{v})\} \quad \text{for all} \quad \mathbf{v} \in V, \\ b(\theta_\varepsilon, \phi) &= \mathscr{R}_{L\varepsilon}\{d(\mathbf{u}_\varepsilon, \phi) + f(\theta_\varepsilon, \phi)\} \quad \text{for all} \quad \phi \in Q. \end{aligned} \tag{2.23}$$

Here $(\cdot, \cdot)$ denotes the L_2-inner product, and

$$a: V \times V \to \mathbb{R}, \qquad a(\mathbf{u}, \mathbf{v}) = 2 \int_\Omega \mathbf{D}(\mathbf{u}) \cdot \mathbf{D}(\mathbf{v})\, dx,$$

$$b: Q \times Q \to \mathbb{R}, \qquad b(\theta, \phi) = \int_\Omega \nabla\theta \cdot \nabla\phi\, dx + \int_{\Gamma_k} k\theta\phi\, ds,$$

$$c: Q \times V \to \mathbb{R}, \qquad c(\theta, \mathbf{v}) = -\int_\Omega \phi \mathbf{v} \cdot \mathbf{g}\, dx,$$

$$d: V \times Q \to \mathbb{R}, \qquad d(\mathbf{v}, \phi) = -\int_\Omega \phi \mathbf{v} \cdot \nabla T\, dx,$$

$$e: V \times V \to \mathbb{R}, \qquad e(\mathbf{u}, \mathbf{v}) = -\beta \int_\Omega [(\nabla \mathbf{U})\, \mathbf{u} + (\nabla \mathbf{u})\, \mathbf{U}] \cdot \mathbf{v}\, dx,$$

$$f: Q \times Q \to \mathbb{R}, \qquad f(\theta, \phi) = -\beta \int_\Omega (\nabla\theta \cdot \mathbf{U})\, \phi\, dx,$$

are all bilinear forms. Equations (2.23) may be written more compactly if we define the bilinear forms

$$A: \bar{V} \times \bar{V} \to \mathbb{R}, \qquad A(\bar{u}, \bar{v}) = a(\mathbf{u}, \mathbf{v}) + b(\theta, \phi)$$

and (2.24)

$$B: \bar{V} \times \bar{V} \to \mathbb{R}, \qquad B(\bar{u}, \bar{v}) = c(\theta, \mathbf{v}) + e(\mathbf{u}, \mathbf{v}) + d(\mathbf{u}, \phi) + f(\theta, \phi).$$

Then (2.23) becomes: find $\bar{u}_\varepsilon = (\mathbf{u}_\varepsilon, \theta_\varepsilon)$ and $\mathscr{R}_{L\varepsilon}$ such that

$$A(\bar{u}_\varepsilon, \bar{v}) + \varepsilon^{-1}(\operatorname{div} \mathbf{u}_\varepsilon, \operatorname{div} \mathbf{v}) = \mathscr{R}_{L\varepsilon}\, B(\bar{u}_\varepsilon, \bar{v}) \qquad \text{for all} \quad \bar{v} = (v, \phi) \in \bar{V}. \tag{2.25}$$

It is straightforward to show that solutions of (2.25) are also solutions of the penalised form of (2.22) (with $2.22)_3$ removed and ∇p replaced by $-\varepsilon^{-1}$ div $\mathbf{u}_\varepsilon$) and vice versa, assuming the solutions to have the requisite smoothness (Carey and Oden [2]).

If $A(\cdot, \cdot)$ and $B(\cdot, \cdot)$ are symmetric then the eigenvalues of (2.25), if they exist, are real. Generally, though, $B(\cdot, \cdot)$ is not symmetric, as can be easily seen from $(2.24)_2$. For the general case there is no theory which gives necessary and sufficient conditions for the existence of real eigenvalues of (2.25), though solutions for particular cases or approximate solutions indicate that the system (2.25) does, in general, possess real eigenvalues (see Joseph [12]).

It is expected that $\bar{u}_\varepsilon \to \bar{u}$, $\mathscr{R}_{L\varepsilon} \to \mathscr{R}_L$, and $p_\varepsilon \to p$ as $\varepsilon \to 0$, where $p_\varepsilon = -\varepsilon^{-1}$ div $\mathbf{u}_\varepsilon$. Proof of convergence is complicated, though, by the fact that $B(\cdot, \cdot)$ is unsymmetric, and it is not pursued, though numerical experiments support the conjecture.

Global Stability

We start by defining the energy at time t, $E(t, \lambda, \bar{v}(t))$, of a disturbance $\bar{v}(t)$, corresponding to a given value of $\lambda > 0$, a coupling parameter, by

$$E(t, \lambda, \bar{v}(t)) = \tfrac{1}{2} \int_\Omega |\mathbf{v}(t)|^2 + \lambda \text{ Pr } \phi^2(t)\, dx. \tag{2.26}$$

We assume that $\mathbf{v}(t) \in W = \{\mathbf{w} \in V\colon \text{div } \mathbf{w} = 0\}$, that $\phi(t) \in Q$ and $\bar{W} = W \times Q$, and that $E(t, \lambda, \cdot)\colon \bar{W} \to \mathbb{R}$. We adopt the conventional definitions of global stability: the solution $(\mathbf{U}, T, P)$ is globally stable if the energy of the disturbance goes to zero as $t \to \infty$. The time rate of change of energy is easily shown to be given by

$$\frac{d}{dt} E(t, \lambda, \bar{u}(t)) = -J_\lambda(\bar{u}(t)) + \mathscr{R}\, I_\lambda(\bar{u}(t)), \tag{2.27}$$

assuming that $\bar{u}(t)$ is a solution of (2.13)–(2.18), where

$$J_\lambda\colon \bar{W} \to \mathbb{R}, \qquad J_\lambda(\bar{v}) = \int_\Omega 2\mathbf{D}(\mathbf{v}) \cdot \mathbf{D}(\mathbf{v}) + \lambda\, |\nabla\phi|^2\, dx + \int_{\Gamma_k} k\phi^2\, ds, \tag{2.28}$$

$$I_\lambda\colon \bar{W} \to \mathbb{R}, \qquad I_\lambda(\bar{v}) = -\int_\Omega \beta\mathbf{v} \cdot \mathbf{L}(t)\,\mathbf{v} + \phi\mathbf{v} \cdot (\mathbf{g} + \lambda\nabla T(t))\, dx, \tag{2.29}$$

and $\mathbf{L}(t) = \frac{1}{2}(\nabla\mathbf{u}(t) + (\nabla\mathbf{u}(t))^T)$. The significance of (2.27) follows from the following theorem [20].

THEOREM 2.1. *Suppose that J is $\bar{V}$-elliptic, that is, there exists a constant $m > 0$ such that*

$$J(\bar{v}) \geq m\, \|\bar{v}\|^2_{\bar{V}}.$$

Suppose further that there are constants $\lambda_1, \lambda_2 > 0$ *and* $\Pr_1, \Pr_2 > 0$ *such that* $0 < \lambda_1 \leqslant \lambda \leqslant \lambda_2$ *and* $0 < \Pr_1 \leqslant \Pr \leqslant \Pr_2$. *Set*

$$\mathscr{R}_\lambda = \inf\{J_\lambda(\bar{w})/I_\lambda(\bar{w}) \colon \bar{v} \in \bar{W}\}. \tag{2.30}$$

Then there is a constant $\gamma > 0$ *such that*

$$E(t, \lambda, \bar{u}(t)) \leqslant E(0, \lambda, \bar{u}(0)) \exp\left\{-\int_0^t \left(1 - \frac{\mathscr{R}(s)}{\mathscr{R}_\lambda}\right) \gamma \, ds\right\}, \tag{2.31}$$

provided that $\mathscr{R}(s) < \mathscr{R}_\lambda$ *for* $0 \leqslant s \leqslant t$. *If* $\mathscr{R}(t) < \mathscr{R}_\lambda$ *for all* t *then* $E(t, \lambda, \bar{u}(t)) \to 0$ *as* $t \to \infty$ *and the flow is asymptotically stable.*

According to Theorem 2.1, if we can find a number $\mathscr{R}_\lambda \in (0, \infty)$ defined by (2.30), then global asymptotic stability is guaranteed when $\mathscr{R} < \mathscr{R}_\lambda$. The exercise may be repeated for different values of λ, in order to find the optimum value of λ, i.e., the value of λ which maximises $\mathscr{R}_\lambda$. Suppose that this maximum is $\mathscr{R}_{\max}$: then we seek $\mathscr{R}_{\max} > 0$, defined by

$$\mathscr{R}_{\max} = \sup_{\lambda > 0} \inf_{\bar{w} \in \bar{W}} \frac{J_\lambda(\bar{w})}{I_\lambda(\bar{w})}. \tag{2.32}$$

The question of existence of such a number is partially resolved in the following result (Shir and Joseph [20], Reddy [17]).

THEOREM 2.2. (a) *For each* $\lambda > 0$ *there exists a minimiser* $\mathscr{R}_\lambda$, *defined by* (2.30). *This minimiser is characterised by the least eigenvalue of the variational eigenvalue problem*

$$\bar{u} \in \bar{W}, \qquad J'_\lambda(\bar{u}) = \mathscr{R}_\lambda I'_\lambda(\bar{u}), \tag{2.33}$$

where $J'(\bar{u}) \colon \bar{W} \to \mathbb{R}$ *is the Gateaux derivative of* J *at* $\bar{u}$ ($I'(\bar{u})$ *is defined similarly*).

(b) *There exists a unique* $\hat{\lambda} < \infty$ *and* $\mathscr{R}_{\max} < \infty$ *such that*

$$\mathscr{R}_{\max} = \mathscr{R}_{\hat{\lambda}} \geqslant \mathscr{R}_\lambda \qquad \text{for all } \lambda \in \mathbb{R}.$$

Remark. The proof that $\hat{\lambda} > 0$ is an open problem, a point not made clear by Shir and Joseph [20]. We also observe, from (2.31), that if $\mathscr{R}_\lambda < 0$, then asymptotic stability for arbitrarily large Rayleigh numbers is guaranteed.

Both the functionals $J_\lambda(\cdot)$ and $I_\lambda(\cdot)$ are Gateaux-differentiable, and (2.33) is equivalent to the following problem: find $\bar{u} \in \bar{W}$ and $\mathscr{R}_\lambda \in \mathbb{R}$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \mathscr{R}_\lambda \{ g(\theta, \mathbf{v}) + h(\mathbf{u}, \mathbf{v}) \}, \qquad \mathbf{v} \in W, \\ \lambda\, b(\theta, \phi) &= \mathscr{R}_\lambda\, g(\phi, \mathbf{u}), \qquad \phi \in Q, \end{aligned} \tag{2.34}$$

where the bilinear forms $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ have been defined by

$$g: Q\times V\to\mathbb{R},\qquad g(\phi,\mathbf{v})=-\tfrac{1}{2}\int_\Omega(\mathbf{g}+\lambda\nabla T(t))\cdot\phi\mathbf{v},$$

$$h: V\times V\to\mathbb{R},\qquad h(\mathbf{u},\mathbf{v})=-\beta\int_\Omega \mathbf{u}\cdot\mathbf{L}(t)\,\mathbf{v}.$$

As with (2.23) we may write Eqs. (2.34) more compactly if we define bilinear forms $\bar{A}(\cdot,\cdot)$ and $\bar{B}(\cdot,\cdot)$ by

$$\begin{aligned}&\bar{A}: \bar{V}\times\bar{V}\to\mathbb{R},\qquad \bar{A}(\bar{u},\bar{v})=a(\mathbf{u},\mathbf{v})+\lambda\, b(\theta,\phi),\\ &\bar{B}: \bar{V}\times\bar{V}\to\mathbb{R},\qquad \bar{B}(\bar{u},\bar{v})=g(\phi,\mathbf{u})+g(\theta,\mathbf{v})+h(\mathbf{u},\mathbf{v}).\end{aligned}\tag{2.35}$$

Indeed, we have

$$\bar{A}(\bar{u},\bar{v})=J'_\lambda(\bar{u})\,\bar{v},$$
$$\bar{B}(\bar{u},\bar{v})=I'_\lambda(\bar{u})\,\bar{v}.$$

Then the problem is: find $\bar{u}\in\bar{W}$ and $\mathscr{R}_\lambda\in\mathbb{R}$ such that

$$\bar{A}(\bar{u},\bar{v})=\mathscr{R}_\lambda\,\bar{B}(\bar{u},\bar{v}),\qquad \bar{v}\in\bar{W}.\tag{2.36}$$

We note that, unlike in problem (2.25), both $\bar{A}$ and $\bar{B}$ are *symmetric* bilinear forms, so that $\mathscr{R}_\lambda$ is always *real*. The optimal value $\mathscr{R}_{\max}$ of $\mathscr{R}_\lambda$ is found from elementary calculus, by setting $d\mathscr{R}_\lambda/d\lambda=0$.

The problem may be posed on the larger space $\bar{V}$ if we introduce the pressure p, a Lagrange multiplier, and consider instead the problem of finding $\bar{u}\in\bar{V}$, $p\in P$, and $\mathscr{R}_\lambda\in\mathbb{R}$ such that

$$\begin{aligned}\bar{A}(\bar{u},\bar{v})-(p,\operatorname{div}\mathbf{v})&=\mathscr{R}_\lambda\bar{B}(\bar{u},\bar{v})\qquad &\bar{v}\in\bar{V},\\ (q,\operatorname{div}\mathbf{u})&=0 &q\in P,\end{aligned}\tag{2.37}$$

where

$$P=\left\{q\in L_2(\Omega):\int_\Omega q\,dx=0\right\}\tag{2.38}$$

with norm

$$\|q\|_0=\left\{\int_\Omega q^2\,dx\right\}^{1/2}.\tag{2.39}$$

Furthermore, we may remove the pressure as a variable by introducing a penalisation (Carey and Oden [2]), that is, by introducing a small compressibility and approximating the pressure by $p_\varepsilon = -\varepsilon^{-1} \operatorname{div} \mathbf{u}_\varepsilon$. Then we seek $\mathbf{u}_\varepsilon \in V$, $\theta_\varepsilon \in Q$, and $\mathscr{R}_{\lambda\varepsilon} \in \mathbb{R}$ such that

$$\bar{A}(\bar{u}_\varepsilon, \bar{v}) + \varepsilon^{-1}(\operatorname{div} \mathbf{u}_\varepsilon, \operatorname{div} \mathbf{v}) = \mathscr{R}_{\lambda\varepsilon} \bar{B}(\bar{u}_\varepsilon, \bar{v}), \qquad \bar{v} \in \bar{V}. \tag{2.40}$$

The relationship of (2.40) to (2.37) or (2.36) is given in the following result (Reddy [17]).

THEOREM 2.3. *Suppose there is a constant* $\alpha > 0$ *such that*

$$\alpha \|q\|_0 \leqslant \sup_{\mathbf{v} \in V/\{0\}} \frac{|(q, \operatorname{div} \mathbf{v})|}{\|\mathbf{v}\|_V} \qquad \text{for all } q \in P.$$

Then there is a constant $C > 0$, *independent of* ε, *such that*

$$|\mathscr{R}_{\lambda\varepsilon} - \mathscr{R}_\lambda| \leqslant C\varepsilon.$$

In other words, the solution to the penalised problem converges to that of the original problem as $\varepsilon \to 0$.

We observe that problems (2.25) and (2.40) coincide when the basic solution has zero velocity ($\mathbf{U} = \mathbf{0}$), $\lambda = 1$, and $\nabla T = \mathbf{g}$. This is in fact the situation for the classical Bénard problem (Joseph [13]): here $\mathscr{R}_{1\varepsilon} = \mathscr{R}_{\mathrm{L}\varepsilon}$, and so we have a *necessary and sufficient* condition for stability: the flow is globally stable if and only if $\mathscr{R} < \mathscr{R}_{\mathrm{L}\varepsilon} = \mathscr{R}_{1\varepsilon}$.

3. FINITE ELEMENT APPROXIMATIONS

We are interested in constructing finite element approximations of problems (2.25) and (2.40), and to this end define a family of finite-dimensional subspaces $\{\bar{V}_h\}_{h \in (0,1)}$ of $\bar{V}$, using conforming finite elements. We assume that Ω is polygonal, and that $\{\bar{V}_h\}$ is generated by regular refinements of the mesh $\bigcup_{e=1}^{E} \bar{\Omega}_e = \bar{\Omega}$; here h is the mesh parameter. The approximation of the penalised problems are:

(a) *linear stability* find $\bar{u}_\varepsilon^h \in \bar{V}_h$ and $\mathscr{R}_{\mathrm{L}\varepsilon}^h \in \mathbb{R}$ such that

$$A(\bar{u}_\varepsilon^h, \bar{v}^h) + \varepsilon^{-1} I[(\operatorname{div} \mathbf{u}_\varepsilon^h)(\operatorname{div} \mathbf{v}^h)] = \mathscr{R}_{\mathrm{L}\varepsilon}^h B(\bar{u}_\varepsilon^h, \bar{v}^h) \qquad \text{for all} \quad \bar{v}^h \in \bar{V}_h \tag{3.1}$$

and

(b) *global stability* find $\bar{u}_\varepsilon^h \in \bar{V}_h$ and $\mathscr{R}_{\lambda\varepsilon}^h \in \mathbb{R}$ such that

$$\bar{A}(\bar{u}_\varepsilon^h, \bar{v}^h) + \varepsilon^{-1} I[(\operatorname{div} \mathbf{u}_\varepsilon^h)(\operatorname{div} \mathbf{v}^h)] = \mathscr{R}_{\lambda\varepsilon}^h \bar{B}(\bar{u}_\varepsilon^h, \bar{v}^h) \qquad \text{for all} \quad \bar{v}^h \in \bar{V}_h. \tag{3.2}$$

Here $I[(\cdot)(\cdot)]$ denotes numerical quadrature: for f, $g \in C(\bar{\Omega})$,

$$I[fg] = \sum_{e=1}^{E} \left(\sum_{k=1}^{G} w_k^e f(\mathbf{x}_k^e) g(\mathbf{x}_k^e) \right), \tag{3.3}$$

where $\mathbf{x}_k^e$ and w_k^e are respectively quadrature points and weights in element e. For reasons of stability we are generally forced to underintegrate the term involving ε^{-1} (Carey and Oden [2]), and so we do not use the conventional L_2-inner product in (3.1) and (3.2). The choice of integration scheme (3.3) defines implicitly a finite-dimensional counterpart P_h of the space P of pressures.

Once again we are concerned with the existence of solutions to (3.1) and (3.2), and with the question of convergence of finite element approximations, Since both problems amount to finite-dimensional (matrix) eigenvalue problems we are assured of the existence of N eigenvalues and N corresponding eigenfunctions ($N = \dim \bar{V}_h$). Problem (3.1) has complex eigenvalues, in general, though it will have at least one real eigenvalue if N is odd. Problem (3.2) has an increasing sequence of real eigenvalues. But as in the case of the continuous problems, there are no guarantees that the smallest eigenvalues will be positive, since $\bar{A}$ leads to a positive-definite matrix, but not $\bar{B}$.

For the global stability problem we have the following convergence result (Reddy [17]).

THEOREM 3.1. *For $\bar{u} \in (H^2(\Omega))^{N+1}$ and $p \in H^1(\Omega)$, and for small enough h, there is a constant $C > 0$ independent of h such that, for given $\lambda > 0$,*

$$|\mathscr{R}_{\lambda\varepsilon}^h - \mathscr{R}_\lambda| \leqslant C(h + \varepsilon). \tag{3.4}$$

4. Examples and Numerical Results

We consider a rectangular container of fluid heated from below and internally. The fluid is assumed stationary, so that $\mathbf{U} = \mathbf{0}$. It is assumed that the extension in the z-direction is sufficiently large so that the three-dimensional problem can be reduced to one of two dimensions, in the $x-y$ plane. The width and depth of the layer are denoted by l and d, respectively, and we set the characteristic length L' equal to d.

Let the unit vector $\mathbf{i}$ point in the direction of y increasing. We consider situations in which the temperature gradient, ∇T, and the dimensionless gravity field, $\mathbf{g}$ are parallel vectors. For our problem we take $\mathbf{g}$ as a constant vertical gravity field, so that

$$\nabla T = \mathbf{i}\, dT/dy \qquad \text{and} \qquad \mathbf{g} = -\mathbf{i}$$

(recall that $\mathbf{g}$ is dimensionless). As in Sparrow *et al.* [21] we consider an internal heat source which gives rise to a quadratic temperature distribution which depends only on y, and which may be written in dimensional variables as

$$T = -\tfrac{1}{2}(s/\kappa)\, y^2 + Ay + B, \tag{4.1}$$

where s is the internal heat-source intensity and κ is the thermal conductivity. If we define the heat-source parameter H_s by

$$H_s = \tfrac{1}{2}(s/\kappa)\, d^2/(T_1 - T_2),$$

where T_1 and T_2 are the temperatures at the bottom and top, respectively, of the layer, and choose the characteristic temperature T' such that

$$T' = (T_1 - T_2)[H_s + 1],$$

then the dimensionless temperature gradient is easily shown to be

$$\nabla T = \frac{[H_s(1-2y) - 1]}{[H_s + 1]}\,\mathbf{i}. \tag{4.2}$$

The Bénard Problem

In the Bénard problem the temperature distribution of the motionless state is linear with no internal heat sources present, so that $H_s = 0$ and

$$\nabla T \cdot \mathbf{g} = -\mathbf{i}.$$

The critical stability limits $\mathscr{R}_L$ and $\mathscr{R}_\lambda$ for the linear and global stability theory coincide with $\lambda = 1$. A necessary and sufficient condition for global stability is thus $\mathscr{R} < \mathscr{R}_1 = \mathscr{R}_{\mathrm{L}}$.

The finite element approximation to the eigenvalue problem (3.1) or (3.2) is

$$(\mathbf{K} + 1/\varepsilon\mathbf{H})\,\mathbf{a} = \mathscr{R}_\varepsilon^h\,\mathbf{Ma}, \tag{4.3}$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{H}$ is the matrix arising from the penalty term, $\mathbf{M}$ is the mass matrix, $\mathscr{R}_\varepsilon^h$ is the lowest eigenvalue, and $\mathbf{a}$ is the corresponding eigenvector.

Nine-noded biquadratic elements were used with 3×3 Gauss quadrature for the integrals contributing to the stiffness and mass matrices. For reasons of stability the integrals contributing to the penalty matrix were computed approximately using reduced 2×2 integration; the nine-noded element with reduced integration is only conditionally stable, but in practice it is quite robust provided that the data is sufficiently smooth (see, for example, Oden, Kikuchi, and Song [15]). No problems of instability were encountered in generating the results which follow.

The eigenvalue problem (4.3) was solved by using the subspace iteration method [1]: in this algorithm one computes the projection of the stiffness and mass matrices onto a subspace and then iterates, using the generalised Jacobi technique, to obtain all the eigenvalues of the subspace simultaneously. In this way the size of the problem is considerably reduced. Though it is the least eigenvalue which is of greatest interest here, in many instances the first two eigenvalues were computed (for example, in the results of Figs. 4.1) in order better to locate the transition from one mode to a new mode, as the ratio l/d was increased.

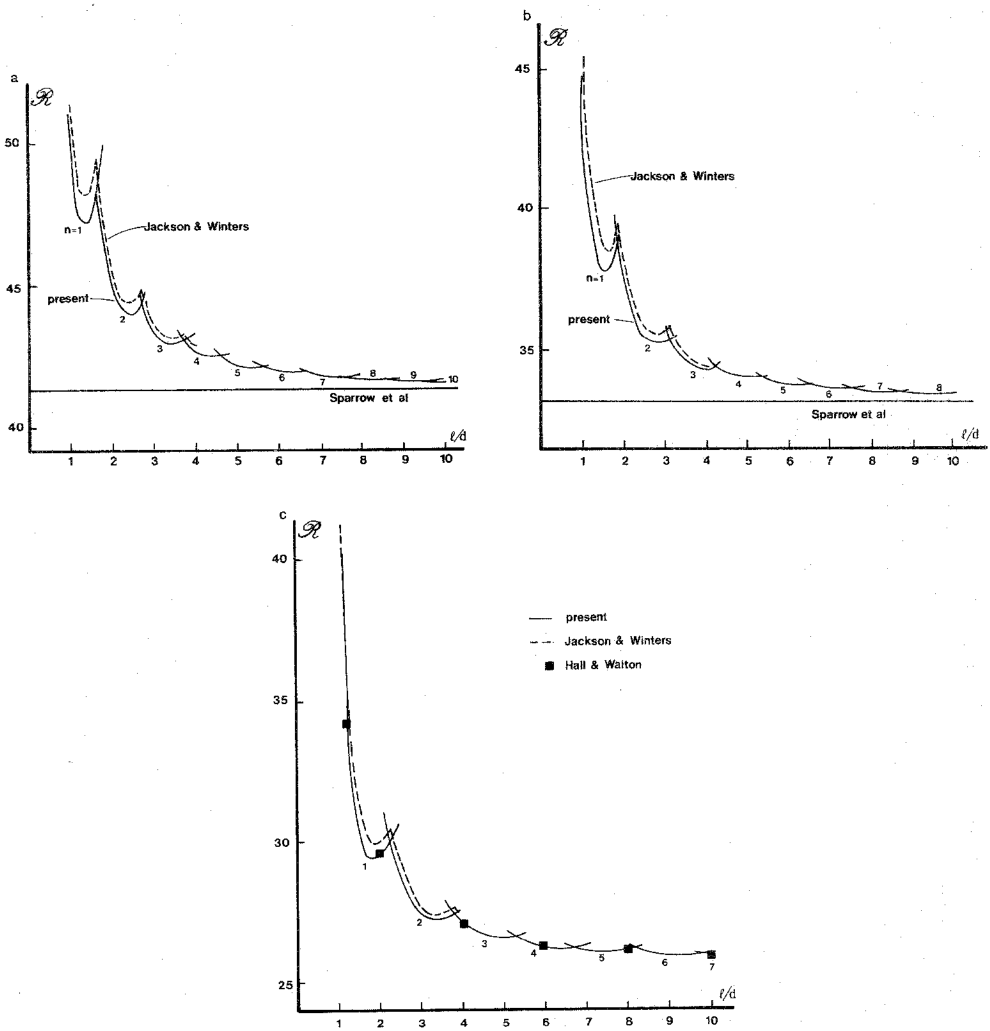


FIG. 4.1. Critical Rayleigh number versus length–depth ratio for Bénard problem, for various velocity boundary conditions at the top and bottom surfaces: (a) rigid top and bottom; (b) free top, rigid bottom; (c) free top and bottom. The number of convection cells at onset of stability is denoted by n.

Extensive numerical experiments indicated that a value of ε of 10^{-4} gave eigenvalues which differed from those corresponding to $\varepsilon = 10^{-3}$ only in the fourth decimal place (for eigenvalues in the range approximately 10 to 50). The value $\varepsilon = 10^{-4}$ was accordingly adopted throughout the study.

Results were obtained for rectangular containers having length–depth ratios in the range 1 to 10. For l/d ratios in the range $1 \leqslant l/d \leqslant 4$ it was found necessary to use square elements of dimensions $\frac{1}{4} \times \frac{1}{4}$ (corresponding to unit depth d) while for ratios in the range $4 \leqslant l \leqslant 10$ elements of depth $\frac{1}{4}$ and length $\frac{1}{2}$ were used (again corresponding to $d = 1$). These relatively fine meshes gave eigenvalues which differed only in the fourth decimal place, when compared with results obtained using elements of dimension $\frac{1}{2} \times \frac{1}{2}$ and $\frac{1}{2} \times 1$, respectively.

All numerical results given in this section are obtained from the penalised finite element problems, but for simplicity we henceforth omit subscripts ε and h.

Consider first a fluid layer with a fixed temperature at both upper and lower bounding surfaces. We investigate the following velocity boundary conditions:

(a) the upper and lower bounding surfaces are both rigid,

(b) the lower surface is rigid while the upper surface is free, and

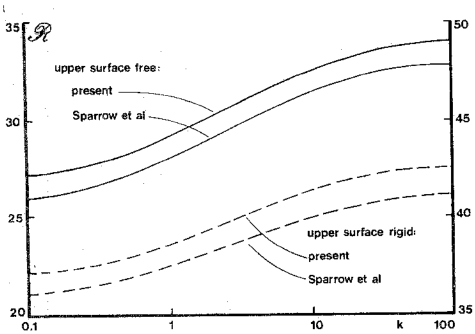
(c) the upper and lower surfaces are free (in the absence of surface tension this condition does not correspond to a real physical situation but may be of theoretical interest).

We require that the side walls be rigid ($\mathbf{u} = \mathbf{0}$) and perfect insulators ($\partial\theta/\partial x = 0$) throughout this study.

The critical Rayleigh numbers for the motionless solution to the Bénard problem with rigid walls on all sides are presented in Fig. 4.1a for l/d in the range 1 to 10. The results for the rigid-free case are presented in Fig. 4.1b and for the free–free case in Fig. 4.1c. Results obtained by Hall and Walton [8], Jackson and Winters [9], and Sparrow *et al.* [21] are presented for comparison. Jackson and Winters computed critical Rayleigh numbers using a mixed finite element method with six-node quadratic triangles to model velocities and temperature, and three-node linear [illegible] Walton [8] for the free–free case and various geometries. Results obtained by Sparrow *et al.* include rigid–rigid and rigid–free horizontal bounding surfaces but are for the case of an infinitely long container.

There is reasonably good corelation with the results of Jackson and Winters [9] and of Hall and Walton [8]. The results of the former were for l/d in the range $1 \leqslant l/d \leqslant 4$, and our results indicate that the trend observed in this range continues, as l/d increases to a value of 10. That is, the envelope of least eigenvalues is a piecewise smooth curve, each smooth section of the curve corresponding to a particular mode number (number of convection cells at onset of instability). The mode number increases discretely with increase in l/d, and the critical Rayleigh number appears to converge (in Figs. 4.1a, b) to the value obtained by Sparrow *et al.* [21] for an infinitely long container. It is worth noting that, for the free–free case, Drazin [5] has in fact shown that $\mathscr{R}(l) > \mathscr{R}(\infty)$ and $\mathscr{R}(l) \rightarrow \mathscr{R}(\infty)$ as $l \rightarrow \infty$, where $\mathscr{R}(l)$ is the critical Rayleigh number for a container of length l, and $\mathscr{R}(\infty)$ for a container of infinite length (see also Hall and Walton [8]).

We now investigate the stability of a fluid for a broad range of temperature boundary conditions. These include a fixed temperature ($\theta = 0$) and fixed heat flux ($d\theta/dy = 0$) at the lower bounding surface and a general convective exchange at the



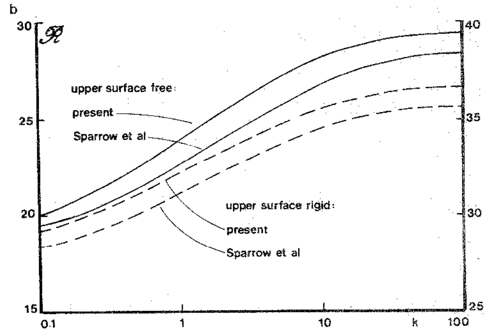


FIG. 4.2. Critical Rayleigh number versus Biot number k at upper surface for the Bénard problem, with $l/d = 4$: (a) fixed temperature at lower surface; (b) fixed heat flux at lower surface.

upper surface ($d\theta/dy + k\theta = 0$). The last condition includes fixed temperature ($k \to \infty$) and fixed heat flux ($k \to 0$) as special cases. We require that the lower bounding surface of the fluid layer be rigid. The upper surface may either be a free surface or a rigid surface.

For each Biot number k there is a critical Rayleigh number below which the motionless state is stable. The critical Rayleigh numbers marking the onset of instability for a fluid layer with $l/d = 4$ are presented in Figs. 4.2. In Fig. 4.2a the lower surface is at a fixed temperature and in Fig. 4.2b the lower surface is at a fixed heat flux. Also shown are the results obtained by Sparrow *et al.* [21] for an infinitely long fluid layer. On each of the graphs there are two curves; the curve corresponding to the free upper surface is referred to the left-hand ordinate scale; the curve corresponding to a rigid upper surface is referred to the right-hand ordinate scale.

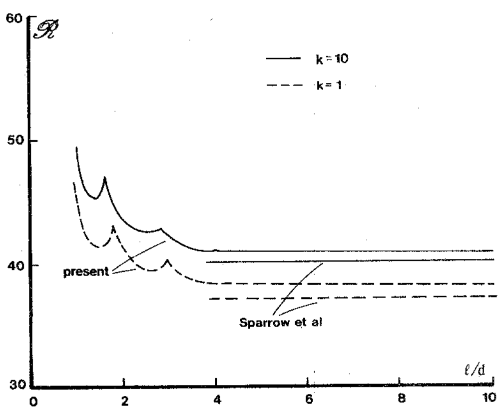


FIG. 4.3. Critical Rayleigh number versus length–depth ratio for the Bénard problem, with fixed temperature at the lower surface, for $k = 1$ and $k = 10$ at upper surface.

From Figs. 4.2a and b it is seen that, for a given velocity boundary condition at the upper surface, the critical Rayleigh number increases monotonically with increasing Biot number k. Thus, the most stable situation corresponds to a fixed temperature ($k \to \infty$), a trend which is in accord with the result for an infinitely long container (Joseph and Shir [14]).

To show convergence to the results of Sparrow *et al.* as $l \to \infty$ for the rigid–rigid case with fixed temperature at the lower surface, a plot of critical Rayleigh number versus l/d is shown in Fig. 4.3. There are two curves; one corresponds to an upper surface with $k = 1$ and the other corresponds to an upper surface with $k = 10$. Both curves approach the limits obtained by Sparrow *et al.* as l/d increases.

Non-linear Temperature Distribution

We now investigate how the stability of a fluid in the motionless state is affected by a variation from linearity of the temperature distribution. The temperature distribution is nonlinear due to an internal heat source in the fluid layer. The temperature gradient is given by (4.2): when $H_s = 0$ we recover the case of linear temperature distribution (the Bénard problem), so that the magnitude of H_s is a rough measure of the degree of nonlinearity. When $H_s \neq 0$ the critical stability limits for the global and linear theory, $\mathscr{R}_\lambda$ and $\mathscr{R}_\mathrm{L}$, do not coincide, and for $\mathscr{R}$ in the range $\mathscr{R}_\lambda < \mathscr{R} < \mathscr{R}_\mathrm{L}$ solutions exist whose energy does not decay even though the stability criterion of the linear theory is satisfied. Such conditions are called subcritical (Joseph [13]).

In order to compare results obtained in this section with those of the previous section we divide the critical Rayleigh number corresponding to a nonlinear temperature distribution by a factor $(H_s + 1)^{1/2}$. This is necessary since the critical Rayleigh number corresponding to a linear temperature distribution is given by

$$\mathscr{R} = \left(\frac{\alpha g\, d^3\, T'}{\kappa \nu}\right)^{1/2},$$

where $T' = (T_1 - T_2)$ is the temperature difference across the fluid layer. However, for the case when we have a nonlinear temperature distribution, T' has been defined by

$$T' = (T_1 - T_2)(H_s + 1).$$

Thus, for $H_s > 0$ we have

$$\mathscr{R} = \left(\frac{\alpha g d^3\, T'}{\kappa \nu (H_s + 1)}\right)^{1/2}.$$

We first calculate the critical values of the Rayleigh number for the linear stability theory. Consider fluid layers with $l/d = 1$ and 10, and the following

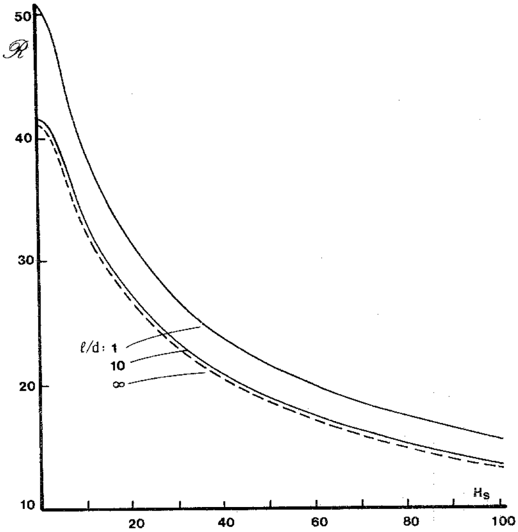


FIG. 4.4. Critical Rayleigh number versus heat-source parameter, according to linear theory.

boundary conditions: horizontal bounding surfaces both rigid and isothermal, and vertical bounding surfaces rigid and perfect insulators. The critical Rayleigh numbers, $\mathcal{R}_L$, are shown in Fig. 4.4 for values of heat-source parameter H_s in the range 1 to 100. The results obtained by Sparrow *et al.* [21] for an infinitely long layer are also shown for comparison. The critical Rayleigh number for $H_s = 1$ is very close to that corresponding to a linear temperature distribution and, as H_s increases the critical Rayleigh number decreases monotonically. Thus, the nonlinear temperature has a destabilising effect on the fluid. Comparing the three sets of results, we see that the curves differ by an almost uniform amount and we expect that this will approach zero as the ratio l/d approaches infinity.

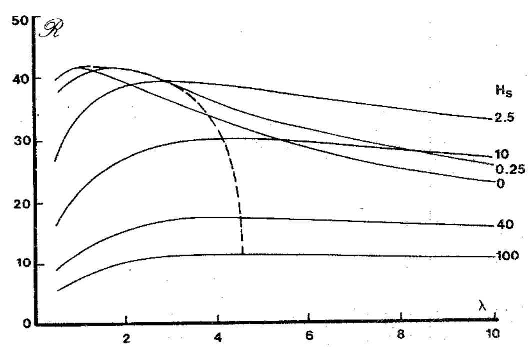


FIG. 4.5. Critical Rayleigh number versus coupling parameter λ, with $l/d = 10$, according to global theory.

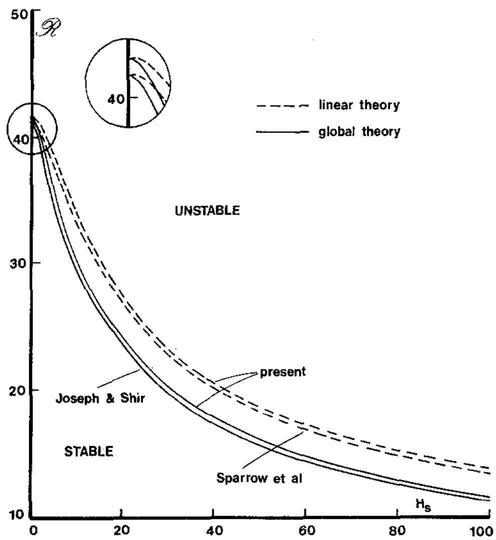


FIG. 4.6. Critical Rayleigh number versus heat-source parameter, with $l/d=10$, for the linear and global theories.

We next obtain critical values of the Rayleigh number for the global stability theory. For this theory stability is guaranteed if $\mathscr{R} < \mathscr{R}_\lambda$ for a fixed value of H_s and fixed $\mathscr{R} > 0$. A set of minimum eigenvalues $\mathscr{R}_\lambda$ are found for different λ keeping H_s fixed. The value $\mathscr{R}_{\text{max}}$ of $\mathscr{R}$ which produces the maximum value of $\mathscr{R}_\lambda$ determines the critical stability limit $\mathscr{R}_{\text{max}}$. The variation of $\mathscr{R}_\lambda$ with λ is given in Fig. 4.5 for values of H_s in the range 0 to 100, for the case $l/d=10$. The dashed line gives the optimal values $\mathscr{R}_{\text{max}}$ corresponding to the range of values of H_s. In Fig. 4.6 the critical Rayleigh numbers, $\mathscr{R}_{\text{L}}$ and $\mathscr{R}_{\text{max}}$, of the linear and global theory are compared, also for $l/d=10$. When $H_s=0$, $\mathscr{R}_{\text{L}}=\mathscr{R}_{\text{max}}$, and no subcritical instabilities exist. For $H_s>0$ the critical Rayleigh numbers for the energy theory are slightly less than those given by the linear theory, the difference increasing from zero with the magnitude of the heat-source intensity. The curve corresponding to the linear theory defines a boundary above which the flow is certainly unstable. The curve corresponding to the energy theory defines a boundary below which the flow is certainly stable. The region between these two curves is open to subcritical instabilities. Results obtained by Joseph and Shir [14] and Sparrow *et al.* [21] for the infinitely long container are also reproduced.

Figure 4.6 indicates that variation in critical values of Rayleigh number with length–depth ratio may not be very marked. This is further borne out by the results shown in Fig. 4.7, where the critical Rayleigh number is plotted against l/d. It should be noted that these curves, like those in Fig. 4.1, are piecewise smooth, but beyond a value of $l/d=4$ it is not possible to show their piecewise nature on a graph of this scale.

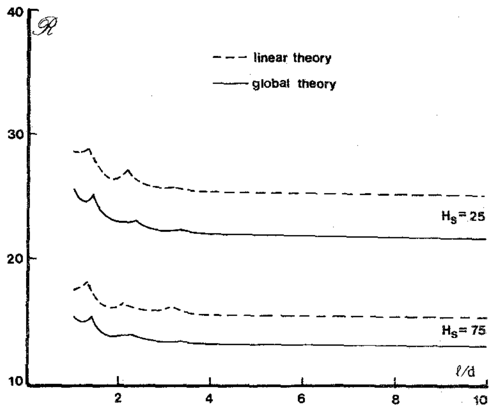


FIG. 4.7. Critical Rayleigh number versus length–depth ratio for linear and global theories.

5. CONCLUDING REMARKS

The penalty-finite element method is evidently well suited to the generation of approximate results for eigenvalue problems arising in fluid stability. It is stable (provided one chooses the element and reduced integration scheme carefully), economical (in the sense that pressure is not present as a variable), accurate, and rests on a firm theoretical basis. A number of issues which have not been addressed here could very well be resolved using this method: for example, problems involving more complex domains, and three-dimensional problems. These must, however, await future study.

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